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The Renormalization Group Method in Modelling of Mechanical Properties of Heterogeneous Polymeric Materials

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SUMMARY

The renormalization group technique, developed in the mean field theory, is applied here as a method for calculating of elastic or viscoelastic properties of heterogeneous materials with complex structures. The method consists in averageing of properties of small volume elements of the heterogeneous system by step-wise application of a recurence relation which renormalizes properties of the group of neighbouring cells to properties of a new cell of larger dimensions. The recurence relations are derived on the basis of the "packet" model - a group of eight elemental cells.

INTRODUCTION

Models applied for predicting mechanical properties of heterogeneous (multi-phase) materials can be classified into various groups : (1) the simple mechanical coupling models which provide a phenomenological representation of the properties-composition dependences, (2) "self-consitent" models based on analysis of mechanical behaviour of representative inclusion in a homogeneous body having the properties of macroscopic composite, and (3) models predicting limits or bounds of mechanical response - based on the mechanical analysis of some idealized structures. Models of all these groups have been reviewed for example by Dickie (1). Recently, a new "packet" model has been proposed (2) which does not fall into any of the above groups. It can be recognized that this model is based on the concept which is very similar to that of renormalization group method (3) developed in the mean field theory. This paper will discuss modelling of mechanical properties of heterogeneous materials on the basis of the renormalization group method as well as it will show a generalized solution related to the "packet" model.

The Renormalization Group Method

Many of the physical problems demand consideration of a large number of independent parameters or variables of the system, while most of the theoretical problems may be succesfully solved only when the number of variables is considerably reduced. The above remark is closely related to the problem of description or mechanical properties of heterogeneous bodies. The reduction of

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variables is usually made by considering a small part (volume element) of the system. On the basis of an analysis of this part, regarded as representative element, the conclusions are drawn about the properties of the macroscopic body. It is, for example, the method on which the "self-consistent" models are based. The question arises, however, which part of the system may be regarded as the representative one. Minimal size of it depends on the system and its state and has to be related to the correlation radius of local structure and the range of interactions. If the correlation radius is small i.e., for example of the order of the inclusion size, the "self-consistent" method seems to provide a reasonable approximation of mechanical properties of macroscopic system. In many real structured heterogeneous systems correlation extends however, over large distances and on various levels of structural order. It is clear that in such cases it is not possible to apply methods which assume direct relations between properties of microscopic elements and properties of macroscopic body and which are consequently valid only for systems in which the correlation radius is small.

The method which can provides solutions of more complicated problems is the renormalization group method. It appeared especially succesfull in the theory of critical phenomena (3, 4). The approximation made in the renormalization group method is based on the reduction of independent variables too, however, in this method it is made not by simple limitation of considerations to a small element of the system, but by a step-wise reduction of dimensions of the element from the macroscopic size to the size of the order of the smallest correlation radius.

The procedure of renormalization when applied to mechanical properties of heterogeneous systems can be formulated as follows. The macroscopic system is divided into cubic cells of dimenions comparable with the smallest size of heterogeneities. It ensures that each cell contains with some geometrical approximation, only one homogeneous component. Mechanical properties of cells are therefore known if only the properties of individulal components of the system are available. Considering at first a short range mechanical interactions between cells we can build up the cells into groups of nearest neighbours. In order to take into account neighbours in three dimensions the smallest group of nearest neighbours has to contain eight elemental cells. In general, the group may contain larger number of cells depending on the range of interactions assumed. The properties of the group are dependent on the properties of individual cells and the parameters describing their mechanical interactions. If we would be able to describe these interactions we could calculate the properties of the group as a whole. The group would consitute in this way a new renormalized cell with known properties. If we repeat the renormalization procedure once again we get again new larger cells with properties dependent on properties and interactions of a large number of elemental cells. Fig. 1 illustrates schematically three steps of renormalization. In each renormalization step the number of variables describing a variety of properties and interactions

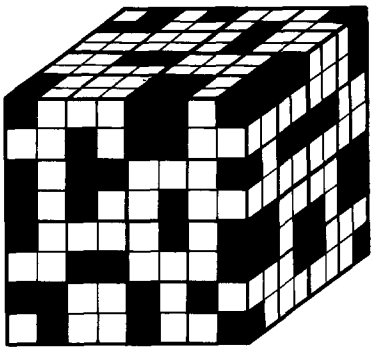

Fig. 1. Illustration of three steps of renormalization in the case of three dimensional heterogeneous system.

of elemental cells is formally reduced to a new variables describing the properties of the group of cells. If we assume that the properties of each elemental cell are described by the modulus matrix E_o we can express the transformation which renormalizes a set of E_{oi} matrices into a new effective modulus matrix E_1 of the group as follows

$$T(E_{oi}) = E_1 \tag{1a}$$

where T denotes the renormalization transformation and i is the number of the cell incorporated in the group. The same renormalization can transform i matrices E_1 into matrix E_2

$$T(E_{1i}) = E_2 \ \ldots \tag{1b}$$

and so on.
If there exist matrices E* for which

$$T(E_i^*) = E^* \tag{2}$$

than they can be regarded as representative for the properties of macroscopic body. The number of renormalization steps leading to matrices E* will describe the size of the smallest cell of the system with the properties of the macroscopic body.

According to the procedure described, if we only know the form of the renormalization transformation T we are able to describe macroscopic properties of the heterogeneous body in terms of the properties of its constituents. To find the form of the transformation T is however not an easy task. An example of such transformation has been proposed for the "packet" model in previous papers (2, 5). Here we would like to present more general form of this transformation considering both normal and shear deformations of cells and considering also noncubic geometrical forms of cells which could better reflect structural peculiarities of heterogeneous systems.

The renormalization transformation based on the "packet" model

The group (or "packet") built up of eight cells is assumed here as a set of nearest neighbours. As described previously (2), the

cells are mechanically coupled in such a way that the adjacent cells deform equally in respective directions. It ensures justification of the continuity requirements inside the group. The cells in the group are denoted by natural indices i, j, k and have in undeformed state the dimensions γ_i, β_j, and α_k along the axes x, y and z respectively. It is assumed that the following conditions are satisfied

$$\sum_{i=1}^{2} \gamma_i = 1, \qquad \sum_{j=1}^{2} \beta_j = 1 \qquad \text{and} \qquad \sum_{k=1}^{2} \alpha_k = 1 \tag{3}$$

which means that the group is treated as a unit element for the actually considered scale.

The strains inside the group are averaged in the following way

$$\begin{aligned}
&\sum_i \gamma_i \varepsilon_x(i,j,k) = \varepsilon_x && \sum_j \beta_j \varepsilon_{yzj} = \varepsilon_{yz} \\
&\sum_j \beta_j \varepsilon_y(i,j,k) = \varepsilon_y && \sum_k \alpha_k \varepsilon_{zyk} = \varepsilon_{zy} \\
&\sum_k \alpha_k \varepsilon_z(i,j,k) = \varepsilon_z && \sum_k \alpha_k \varepsilon_{zxk} = \varepsilon_{zx} \\
&\sum_i \gamma_i \varepsilon_{xyi} = \varepsilon_{xy} && \sum_i \gamma_i \varepsilon_{xzi} = \varepsilon_{xz} \\
&\sum_j \beta_j \varepsilon_{yxj} = \varepsilon_{yx}
\end{aligned} \tag{4}$$

where the shear strain components on the right hand side of equations are the mean strains of the whole group and the strain components on the left hand side are the strains of respective adjacent cells.

It is assumed that the mean value of stress components acting on all cells of the group are equal to respective extrenal stresses acting on the group. This leads to the following equations of the stress balance

$$\sum_{jk} \beta_j \alpha_k \sigma_x(i,j,k) = \sigma_x \quad (a) \qquad \sum_{ik} \gamma_i \alpha_k \sigma_{yz}(i,j,k) = \sigma_{yz} \quad (f)$$

$$\sum_{ki} \alpha_k \gamma_i \sigma_y(i,j,k) = \sigma_y \quad (b) \qquad \sum_{ij} \gamma_i \beta_j \sigma_{zy}(i,j,k) = \sigma_{zy} \quad (g)$$

$$\sum_{ij} \gamma_i \beta_j \sigma_z(i,j,k) = \sigma_z \quad (c) \qquad \sum_{ij} \gamma_i \beta_j \sigma_{zx}(i,j,k) = \sigma_{zx} \quad (h)$$

$$\sum_{jk} \beta_j \alpha_k \sigma_{xy}(i,j,k) = \sigma_{xy} \quad (d) \qquad \sum_{jk} \beta_j \alpha_k \sigma_{xz}(i,j,k) = \sigma_{xz} \quad (i)$$

$$\sum_{ik} \gamma_i \alpha_k \sigma_{yx}(i,j,k) = \sigma_{yx} \quad (e)$$

The properties of every element in the group are given by the constitutive equations in the following from

$$[\sigma_n(i,j,k)]=[E_{mn}(i,j,k)]\,[\varepsilon_m(i,j,k)] \qquad (6)$$

where $[\sigma_n(i,j,k)]$ and $[\sigma_m(i,j,k)]$ are 9-element matrices of stress and strain components respectively, and $[E_{mn}(i,j,k)]$ reptesents the matrix of modulae of the element i,j,k. The form of matrix is dependent on the type of mechanical properties of the cell. The modulae matrix assumed here has the form

$$[E_{mn}(i,j,k)]=\begin{bmatrix} E_{11} & E_{12} & E_{13} & & & \\ E_{12} & E_{22} & E_{33} & & & \\ E_{13} & E_{23} & E_{33} & & & \\ & & & E_{44} & & \\ & & & & E_{55} & \\ & & & & & E_{66} \end{bmatrix} \qquad (6a)$$

which considers the anisotropy of mechanical properties of cells. Introducing the following assignements

$$\varepsilon_x(1,j,k)=\varepsilon_1 \qquad \varepsilon_y(i,2,k)=\varepsilon_4$$
$$\varepsilon_x(2,j,k)=\varepsilon_2 \qquad \varepsilon_z(i,j,1)=\varepsilon_5 \qquad (7)$$
$$\varepsilon_y(i,1,k)=\varepsilon_3 \qquad \varepsilon_z(i,j,k)=\varepsilon_6$$

and combining eqs. (5a-c) and (6) we can write

$$\sum_n P_{1n}\varepsilon_n=\sigma_x \quad (a) \qquad \sum_n P_{4n}\varepsilon_n=\sigma_y \quad (d)$$
$$\sum_n P_{2n}\varepsilon_n=\sigma_x \quad (b) \qquad \sum_n P_{6n}\varepsilon_n=\sigma_z \quad (e) \qquad (8)$$
$$\sum_n P_{3n}\varepsilon_n=\sigma_y \quad (c) \qquad \sum_n P_{6n}\varepsilon_n=\sigma_z \quad (f)$$

where the values of P_{mn} coefficients are :

$$P_{11}=\sum_{kj}\sum\alpha_k\beta_j E_{11}(1,j,k) \qquad P_{21}=0$$

$$P_{12}=0 \qquad P_{22}=\sum_{kj}\sum\alpha_k\beta_j E_{11}(2,j,k)$$

$$P_{13}=\beta_1\sum_k\alpha_k E_{12}(1,1,k) \qquad P_{23}=\beta_1\sum_k\alpha_k E_{12}(2,1,k)$$

$$P_{14}=\beta_2\sum_k\alpha_k E_{12}(1,2,k) \qquad P_{24}=\beta_2\sum_k\alpha_k E_{12}(2,2,k)$$

$$P_{15}=\alpha_1\sum_j\beta_j E_{13}(1,j,1) \qquad P_{25}=\alpha_1\sum_j\beta_j E_{13}(2,j,1)$$

$P_{16}=\alpha_2\sum_j\beta_j E_{13}(1,j,2)$	$P_{26}=\alpha_2\sum_j\beta_j E_{13}(2,j,2)$
$P_{31}=\gamma_1\sum_k\alpha_k E_{21}(1,1,k)$	$P_{41}=\gamma_1\sum_k\alpha_k E_{21}(1,2,k)$
$P_{32}=\gamma_2\sum_k\alpha_k E_{21}(2,1,k)$	$P_{42}=\gamma_2\sum_k\alpha_k E_{21}(2,2,k)$
$P_{33}=\sum\sum_{ki}\alpha_k\gamma_i E_{22}(i,1,k)$	$P_{43}=0$
$P_{34}=0$	$P_{44}=\sum\sum_{ki}\alpha_k\gamma_i E_{22}(i,2,k)$
$P_{35}=\alpha_1\sum_i\gamma_i E_{23}(i,1,1)$	$P_{45}=\alpha_1\sum_i\gamma_i E_{23}(i,2,1)$
$P_{36}=\alpha_2\sum_i\gamma_i E_{23}(i,1,2)$	$P_{46}=\alpha_2\sum_i\gamma_1 E_{23}(i,2,2)$
$P_{51}=\gamma_1\sum_j\beta_j E_{31}(1,j,1)$	$P_{61}=\gamma_1\sum_j\beta_j E_{31}(1,j,2)$
$P_{52}=\gamma_2\sum_j\beta_j E_{31}(2,j,1)$	$P_{62}=\gamma_2\sum_j\beta_j E_{31}(2,j,2)$
$P_{53}=\beta_1\sum_i\gamma_i E_{32}(i,1,1)$	$P_{63}=\beta_1\sum_i\gamma_i E_{32}(i,1,2)$
$P_{54}=\beta_2\sum_i\gamma_i E_{32}(i,2,1)$	$P_{64}=\beta_2\sum_i\gamma_i E_{32}(i,2,2)$
$P_{55}=\sum\sum_{ij}\gamma_i\beta_j E_{33}(i,j,1)$	$P_{65}=0$
$P_{56}=0$	$P_{66}=\sum\sum_{ij}\gamma_i\beta_j E_{33}(i,j,k)$

From eqs. (7a,b and c) we can calculate three strain components in the following form

$$\begin{aligned}\varepsilon_1&=(S_{11}\varepsilon_2+S_{12}\varepsilon_4+S_{13}\varepsilon_6+S_{14}\sigma_x+S_{15}\sigma_y+S_{16}\sigma_z)/S\\ \varepsilon_3&=(S_{21}\varepsilon_2+S_{22}\varepsilon_4+S_{23}\varepsilon_6+S_{24}\sigma_x+S_{25}\sigma_y+S_{26}\sigma_z)/S\\ \varepsilon_5&=(S_{31}\varepsilon_2+S_{32}\varepsilon_4+S_{33}\varepsilon_6+S_{34}\sigma_x+S_{35}\sigma_y+S_{36}\sigma_z)/S\end{aligned} \tag{9}$$

where coefficients S_{mn} are given by determinants

$$S=\begin{vmatrix}P_{11} & P_{13} & P_{15}\\ P_{31} & P_{33} & P_{35}\\ P_{51} & P_{53} & P_{55}\end{vmatrix}$$

$$S_{11}=\begin{vmatrix}-P_{12} & P_{13} & P_{15}\\ -P_{32} & P_{33} & P_{35}\\ -P_{52} & P_{53} & P_{55}\end{vmatrix},\quad S_{12}=\begin{vmatrix}-P_{14} & P_{13} & P_{15}\\ -P_{34} & P_{33} & P_{35}\\ -P_{54} & P_{53} & P_{55}\end{vmatrix},\quad S_{13}=\begin{vmatrix}-P_{16} & P_{13} & P_{15}\\ -P_{36} & P_{33} & P_{35}\\ -P_{56} & P_{53} & P_{55}\end{vmatrix},$$

$$S_{21}=\begin{vmatrix}P_{11}-P_{12} & P_{15}\\ P_{31}-P_{32} & P_{35}\\ P_{51}-P_{52} & P_{55}\end{vmatrix},\quad S_{22}=\begin{vmatrix}P_{11}-P_{14} & P_{15}\\ P_{31}-P_{34} & P_{35}\\ P_{51}-P_{54} & P_{55}\end{vmatrix},\quad S_{23}=\begin{vmatrix}P_{11}-P_{14} & P_{15}\\ P_{31}-P_{36} & P_{35}\\ P_{51}-P_{56} & P_{55}\end{vmatrix},$$

$$S_{31}=\begin{vmatrix}P_{11} & P_{13}-P_{12}\\ P_{31} & P_{33}-P_{32}\\ P_{51} & P_{53}-P_{52}\end{vmatrix},\quad S_{32}=\begin{vmatrix}P_{11} & P_{13}-P_{14}\\ P_{31} & P_{33}-P_{34}\\ P_{51} & P_{53}-P_{54}\end{vmatrix},\quad S_{33}=\begin{vmatrix}P_{11} & P_{13}-P_{16}\\ P_{31} & P_{33}-P_{36}\\ P_{51} & P_{53}-P_{56}\end{vmatrix},$$

$$S_{14}=\begin{vmatrix}P_{33} & P_{35}\\ P_{53} & P_{55}\end{vmatrix},\quad S_{15}=\begin{vmatrix}P_{13} & P_{15}\\ P_{53} & P_{55}\end{vmatrix},\quad S_{16}=\begin{vmatrix}P_{13} & P_{15}\\ P_{33} & P_{35}\end{vmatrix},$$

$$S_{24}=\begin{vmatrix}P_{31} & P_{35}\\ P_{51} & P_{55}\end{vmatrix},\quad S_{25}\begin{vmatrix}P_{11} & P_{15}\\ P_{51} & P_{55}\end{vmatrix},\quad S_{26}=\begin{vmatrix}P_{11} & P_{15}\\ P_{31} & P_{35}\end{vmatrix},$$

$$S_{34}=\begin{vmatrix}P_{31} & P_{33}\\ P_{51} & P_{53}\end{vmatrix},\quad S_{35}=\begin{vmatrix}P_{11} & P_{13}\\ P_{51} & P_{53}\end{vmatrix},\quad S_{36}=\begin{vmatrix}P_{11} & P_{13}\\ P_{31} & P_{33}\end{vmatrix},$$

If the values of ε given by eqs. (9) are introduced to eqs. (8b), (8d) and (8f) the latest may be rearranged in the following form

$$\begin{aligned}
T_{11}\varepsilon_2+T_{12}\varepsilon_4+T_{13}\varepsilon_6 &= T_{14}\sigma_x+T_{15}\sigma_y+T_{16}\sigma_z\\
T_{21}\varepsilon_2+T_{22}\varepsilon_4+T_{23}\varepsilon_6 &= T_{24}\sigma_x+T_{25}\sigma_y+T_{26}\sigma_z \qquad (10)\\
T_{31}\varepsilon_2+T_{32}\varepsilon_4+T_{33}\varepsilon_6 &= T_{34}\sigma_x+T_{35}\sigma_x+T_{36}\sigma_z
\end{aligned}$$

where

$$\begin{aligned}
T_{11}&=P_{22}+(P_{23}S_{21}+P_{25}S_{31})/S & T_{14}&=1-(P_{23}S_{24}+P_{25}S_{34})/S\\
T_{12}&=P_{24}+(P_{23}S_{22}+P_{25}S_{32})/S & T_{15}&=-(P_{23}S_{25}+P_{25}S_{35})/S\\
T_{13}&=P_{26}+(P_{23}S_{23}+P_{25}S_{33})/S & T_{16}&=-(P_{23}S_{26}+P_{25}S_{36})/S\\
T_{21}&=P_{42}+(P_{41}S_{11}+P_{43}S_{31})/S & T_{24}&=-(P_{41}S_{14}+P_{43}S_{34})/S\\
T_{22}&=P_{44}+(P_{41}S_{12}+P_{43}S_{32})/S & T_{25}&=1-(P_{41}S_{15}+P_{43}S_{35})/S\\
T_{23}&=P_{46}+(P_{41}S_{13}+P_{43}S_{33})/S & T_{26}&=-(P_{41}S_{16}+P_{43}S_{36})/S\\
T_{31}&=P_{62}+(P_{61}S_{11}+P_{63}S_{21})/S & T_{34}&=-(P_{61}S_{14}+P_{63}S_{24})/S\\
T_{32}&=P_{64}+(P_{61}S_{12}+P_{63}S_{22})/S & T_{35}&=-(P_{61}S_{15}+P_{63}S_{25})/S\\
T_{33}&=P_{66}+(P_{61}S_{13}+P_{63}S_{23})/S & T_{36}&=1-(P_{61}S_{16}+P_{63}S_{26})/S
\end{aligned}$$

From eqs. (10) one can calculate the strain components

$$\varepsilon_2=A_{41}\sigma_x+A_{42}\sigma_y+A_{43}\sigma_z$$
$$\varepsilon_4=A_{51}\sigma_x+A_{52}\sigma_y+A_{53}\sigma_z \qquad (11)$$
$$\varepsilon_6=A_{61}\sigma_x+A_{62}\sigma_y+A_{63}\sigma_z$$

where

$$A_{41}=\frac{1}{A}\begin{vmatrix} T_{14} & T_{12} & T_{13} \\ T_{24} & T_{22} & T_{23} \\ T_{34} & T_{32} & T_{33} \end{vmatrix}, \quad A_{42}=\frac{1}{A}\begin{vmatrix} T_{15} & T_{12} & T_{13} \\ T_{25} & T_{22} & T_{23} \\ T_{35} & T_{32} & T_{33} \end{vmatrix}, \quad A_{43}=\frac{1}{A}\begin{vmatrix} T_{16} & T_{12} & T_{13} \\ T_{26} & T_{22} & T_{23} \\ T_{36} & T_{32} & T_{33} \end{vmatrix}$$

$$A_{51}=\frac{1}{A}\begin{vmatrix} T_{11} & T_{14} & T_{13} \\ T_{21} & T_{24} & T_{23} \\ T_{31} & T_{34} & T_{33} \end{vmatrix}, \quad A_{52}=\frac{1}{A}\begin{vmatrix} T_{11} & T_{15} & T_{13} \\ T_{21} & T_{25} & T_{23} \\ T_{31} & T_{35} & T_{33} \end{vmatrix}, \quad A_{53}=\frac{1}{A}\begin{vmatrix} T_{11} & T_{16} & T_{13} \\ T_{21} & T_{26} & T_{23} \\ T_{31} & T_{36} & T_{33} \end{vmatrix}$$

$$A_{61}=\frac{1}{A}\begin{vmatrix} T_{11} & T_{12} & T_{14} \\ T_{21} & T_{22} & T_{24} \\ T_{31} & T_{32} & T_{34} \end{vmatrix}, \quad A_{62}=\frac{1}{A}\begin{vmatrix} T_{11} & T_{12} & T_{15} \\ T_{21} & T_{22} & T_{25} \\ T_{31} & T_{32} & T_{35} \end{vmatrix}, \quad A_{63}=\frac{1}{A}\begin{vmatrix} T_{11} & T_{12} & T_{16} \\ T_{21} & T_{22} & T_{26} \\ T_{31} & T_{32} & T_{36} \end{vmatrix}$$

$$A = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix}$$

and the other strain components according to (9) and (11) are

$$\varepsilon_1=A_{11}\sigma_x+A_{12}\sigma_y+A_{13}\sigma_z$$
$$\varepsilon_3=A_{21}\sigma_x+A_{22}\sigma_y+A_{23}\sigma_z \qquad (12)$$
$$\varepsilon_5=A_{31}\sigma_x+A_{32}\sigma_y+A_{33}\sigma_z$$

where

$$A_{11}=(S_{14}+S_{11}A_{41}+S_{12}A_{51}+S_{13}A_{61})/S \qquad A_{22}=(S_{25}+S_{21}A_{42}+S_{22}A_{52}+S_{23}A_{62})/S$$
$$A_{12}=(S_{15}+S_{11}A_{42}+S_{12}A_{52}+S_{13}A_{62})/S \qquad A_{23}=(S_{26}+S_{21}A_{43}+S_{22}A_{53}+S_{23}A_{63})/S$$
$$A_{13}=(S_{16}+S_{11}A_{43}+S_{12}A_{53}+S_{13}A_{63})/S \qquad A_{31}=(S_{34}+S_{31}A_{41}+S_{32}A_{51}+S_{33}A_{61})/S$$
$$A_{21}=(S_{24}+S_{21}A_{41}+S_{22}A_{51}+S_{23}A_{61})/S \qquad A_{32}=(S_{35}+S_{31}A_{42}+S_{32}A_{52}+S_{33}A_{62})/S$$
$$A_{33}=(S_{36}+S_{31}A_{43}+S_{32}A_{53}+S_{33}A_{63})/S$$

Then according to equations (4) we have

$$\varepsilon_x=(\gamma_1A_{11}+\gamma_2A_{41})\sigma_x+(\gamma_1A_{12}+\gamma_2A_{42})\sigma_y+(\gamma_1A_{13}+\gamma_2A_{43})\sigma_z$$
$$\varepsilon_y=(\beta_1A_{21}+\beta_2A_{51})\sigma_x+(\beta_1A_{22}+\beta_2A_{52})\sigma_y+(\beta_1A_{23}+\beta_2A_{53})\sigma_z \qquad (13)$$
$$\varepsilon_z=(\alpha_1A_{31}+\alpha_2A_{61})\sigma_x+(\alpha_1A_{32}+\alpha_2A_{62})\sigma_y+(\alpha_1A_{33}+\alpha_2A_{63})\sigma_z$$

which gives a part of constitutive equation of the group of cells in the form

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix} \begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{bmatrix} \tag{14}$$

where C_{ij} are the compliances of the whole group expressed by respective coefficients in eq. (13) i.e. for example

$$\begin{aligned} C_{11}&=\gamma_1 A_{11}+\gamma_2 A_{41} & C_{12}&=\gamma_1 A_{12}+\gamma_2 A_{42} \ \ldots\\ C_{21}&=\beta_1 A_{21}+\beta_2 A_{51} & C_{22}&=\beta_1 A_{22}+\beta_2 A_{52} \ \ldots\\ &\ldots\ldots\ldots\ldots & &\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{15}$$

In analogous way by solution of sets of eqs. (4), (5) and (6) we get for shear deformation :

$$\begin{aligned} \varepsilon_{yz}&=\Big(\sum_j \frac{\beta_j}{R_{1j}}\Big)\sigma_{yz} & \varepsilon_{xz}&=\Big(\sum_i \frac{\gamma_i}{R_{4i}}\Big)\sigma_{xz}\\ \varepsilon_{zy}&=\Big(\sum_k \frac{\alpha_k}{R_{2k}}\Big)\sigma_{zy} & \varepsilon_{xy}&=\Big(\sum_i \frac{\gamma_i}{R_{5i}}\Big)\sigma_{xy}\\ \varepsilon_{zx}&=\Big(\sum_k \frac{\alpha_k}{R_{3k}}\Big)\sigma_{zx} & \varepsilon_{yx}&=\Big(\sum_j \frac{\beta_j}{R_{6j}}\Big)\sigma_{yx} \end{aligned} \tag{16}$$

where the coefficients R are given by

$$\begin{aligned} R_{1j}&=\sum_{ik}\gamma_i\alpha_k E_{44}(i,j,k) & R_{4i}&=\sum_{jk}\beta_j\alpha_k E_{77}(i,j,k)\\ R_{2k}&=\sum_{ij}\gamma_i\beta_j E_{55}(i,j,k) & R_{5i}&=\sum_{jk}\beta_j\alpha_k E_{88}(i,j,k)\\ R_{3k}&=\sum_{ij}\gamma_i\beta_j E_{66}(i,j,k) & R_{6j}&=\sum_{ik}\gamma_i\alpha_k E_{99}(i,j,k) \end{aligned} \tag{17}$$

Relations (14) and (16) play a role of constitutive equations of the group of eight cells. The compliance constants C_{ij} in these relations can be recalculated to respective modulus ^{ij}constants for the group which are dependent on modulae of cells constituting the group. In this way the solution described by eq. (1) is found in non-explicite form and may be used as a recurence relation in the renormalization procedure.

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